CC-3 : Mathematical Analysis

Unit 1:

A) Short type questions (Marks 5):

1. Prove the fundamental theorem of real analysis.
2. State and prove Archimedean Property.
3. Show that between any two real numbers there exists a rational number.
4. What is completeness property of $\mathbb{R}$? Does the set of rational numbers possess the property?
5. What is cluster point of a set? Find $Cluster(A)$, where $A = (0,1)$.
6. If c_0 is an interior point of A then it must be a cluster point of the set.
7. Define convergence and divergence of real sequence. Show that if a real sequence is convergent then its limit must be unique.
8. Check using definition the convergence of the following sequences (each having 5 marks): i) $a_n = (-1)^n, n \geq 1$; ii) $a_n = \frac{(-1)^n}{n}, n \geq 1$; iii) $a_n = \frac{2n+3}{n+1000}$
9. Define monotonic sequence. Show that a sequence of real numbers $a_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$ is convergent.
10. Suppose $\lim_{n\to\infty} a_n = l$ then $\lim_{n\to\infty} 1/a_n = 1/l$, provided $a_n \neq 0 \; \forall \, n \geq 1 \; \& \; l \neq 0$.
11. Show that a positive term series is convergent if and only if it is bounded.
12. What are absolute convergence and conditional convergence of a real series $\sum_{n\geq1} a_n$. Show that if $\sum_{n\geq1} a_n$ is convergent, where $a_n > 0$, then the series $\sum_{n\geq1} \sqrt{a_n a_{n+1}}$ is also convergent.
13. For a convergent sequence $\sum_{n\geq1} a_n$, show that $\lim_{n\to\infty} a_n = 0$.
14. Show that $\sum_{n\geq1} a_n$ is convergent absolutely if and only if $\sum_{n\geq1} a_n^+$ and $\sum_{n\geq1} a_n^-$ both are convergent where $a_n^+ = \max(a_n, 0)$ and $a_n^- = -\min(a_n, 0)$.
15. State and prove Leibnitz Theorem.

B) Broad type questions (Marks 10):

1. State and prove Squeeze theorem. Check for convergence of a real sequence $a_n = x^n \, ; x \in \mathbb{R}$.
2. Define Cauchy sequence of real numbers with an example. Show that a sequence is a Cauchy sequence if and only if it is convergent.
3. Describe Cauchy's first and second theorem on limit. Give suitable examples. Show that for a real sequence $\{a_n\}$ if $S_n = \sum_{k=1}^{n} a_k \to s \, (\in \mathbb{R})$ then $\frac{1}{n}\sum_{k=1}^{n} k \, a_k \to 0$ as $n \to \infty$.
4. Describe root test and ratio test for convergence of real series. Show with an example that root test is more powerful than ratio test.
5. If a real series is convergent absolutely then show that it is convergent in ordinary sense. Show that $\sum_{n\geq1} a_n$ is convergent conditionally then $\sum_{n\geq1} a_n^+$ and $\sum_{n\geq1} a_n^-$ both are divergent where $a_n^+ = \max(a_n, 0)$ and $a_n^- = -\min(a_n, 0)$.
6. What is rearrangement of a real series? State Riemann's theorem regarding rearrangement of real series. Justify the above with the series $\sum_{n\geq1}(-1)^{n-1}/n$.

Unit 2:

C) Short type questions (Marks 5):

1. Let $f(x) = [x]$, the greatest integer contained in x. Using definition show that $\lim f(x)$ exists at $x = 2.8$ but not at $x = 2$.
2. Consider the following function
$$f(x) = \begin{cases} 1 & if\ x \in \mathbb{Q} \\ 0 & if\ x \notin \mathbb{Q} \end{cases}$$
Show that limit of f does not exist at any $c \in \mathbb{Q}$ (state any result required by you).
3. Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be two functions and c be a cluster point of D such that $\lim_{x\to c} f(x) = l_1$ and $\lim_{x\to c} g(x) = l_2$. Show that using definition (each having 5 marks)
   i) $\lim_{x\to c}\{f(x) \pm g(x)\} = l_1 \pm l_2$
   ii) $\lim_{x\to c} f(x)\, g(x) = l_1 l_2$
   iii) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{l_1}{l_2}$, provided $l_2 \neq 0$.
4. Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be two functions and c be a cluster point of D such that $\lim_{x\to c} f(x) = 0$ and $|g(x)| \leq M < \infty$ for all $x \in D - \{c\}$. Show that using definition $\lim_{x\to c} f(x)\, g(x) = 0$.
5. What do you mean by a locally bounded function? Show that $f(x) = 1/x$ is not bounded on $(0,1)$ but it is locally bounded at every point on $(0,1)$.
6. Show that if $\lim_{x\to c} f(x) = l \in \mathbb{R}$ then f is locally bounded at $x = c$ (a cluster point of the domain of f).
7. Show that if $\lim_{x\to c} f(x) = l \neq 0$, then there is an interval around c such that $f(x) \neq 0$ on the interval.
8. Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be two functions and c be a cluster point of D such that $f(x) \leq g(x)$ for all $x \in D - \{c\}$. Show that $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$.
9. State and prove Squeeze theorem in connection with limit of a real function.
10. Let $f: D \to \mathbb{R}$, be a function with c being the cluster point of D and $\lim_{x\to c} f(x) = l \in \mathbb{R}$. Define $g_n(x) = a_n + b_n\, f(x)$ for $x \in D$ and $n \geq 1$, where $\{a_n\}$ and $\{b_n\}$ are real sequences such that $\lim a_n = a$ and $\lim b_n = b$. Show that using definition
$$\lim_{n\to\infty} \lim_{x\to c} g_n(x) = a + bl.$$
11. Show that a polynomial of degree $n \in \mathbb{N}$ is continuous everywhere (clearly state any results you need to use).
12. Show that if $f: [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ then f is bounded on $[a, b]$.
13. If $f: D \to \mathbb{R}$ is continuous on D then $g = \sqrt{f}$ is also continuous on D provided $f(x) > 0$ for all $x \in D$.
14. Check for uniform continuity of the followings (each having 5 marks)
   i) $f(x) = \frac{1}{x}$ on $[a, \infty)$ with $a > 0$
   ii) $f(x) = x^2$ on $(-b, b)$ with $b > 0$
   iii) $f(x) = \sqrt{x}$ on $[0, \infty)$
15. State Lagrange's Mean Value Theorem. Hence show that if the derivative of a function is positive on a subset of its domain then the function is monotonically increasing on the set.

16. State Lagrange's Mean Value Theorem (MVT). Hence show that if a function, $f: [a, b] \to \mathbb{R}$, satisfies the condition of Lagrange's MVT and the derivative of f is bounded on $[a, b]$ then f is uniformly continuous on $[a, b]$.
17. Derive Taylor's polynomial of degree n generated from
    i) $f(x) = e^x$ about $x_0 = 0$
    ii) $f(x) = \ln(1 + x)$ about $x_0 = 0$
    iii) $f(x) = \ln x$ about $x_0 = 1$
    iv) $f(x) = \sin x$ about $x_0 = 0$
    v) $f(x) = \sin x$ about $x_0 = 0$
18. Describing L'Hospital's rule(s) find the followings
    i) $\lim_{x\to 0+} \frac{\sin x}{\sqrt{x}}$
    ii) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$
    iii) $\lim_{x\to 0} \frac{e^x-1}{x}$
    iv) $\lim_{x\to 0} \frac{\ln(1+x)}{x}$
    v) $\lim_{x\to \infty} \frac{\ln x}{x}$
    vi) $\lim_{x\to \infty} x^2 e^{-x}$
    vii) $\lim_{x\to 0+} x^x$
    viii) $\lim_{x\to \infty} \left(1 + \frac{1}{x}\right)^x$

D) Broad type questions (Marks 10):

1. Show that $\lim_{x\to c} f(x) = l \in \mathbb{R}$, where c is a cluster point of D (the domain of f), if and only if for every sequence $\{x_n\}$ in $D - \{c\}$, $f(x_n) \to l$ as $n \to \infty$. Hence show that $\lim_{x\to 2}[x]$ does not exist, where $[x]$ is the greatest integer contained in x.
2. State and prove Intermediate Value Theorem. Hence show that a polynomial of degree n has a real root if n is odd.
3. Describe different kind of discontinuities of a function with suitable examples.
4. What is Lipschitz function? Show that a Lipschitz function is uniformly continuous. Provide an example, with appropriate justification, of a function which uniformly continuous but not a Lipschitz function.
5. What do you mean by derivative of a function at a point? How you define derivative if the point is a boundary point of its domain. Provide geometric interpretation of the above two situations.
6. Construct three functions (with appropriate justification) such that
    i) The function is nowhere continuous
    ii) The function is continuous at a single point but nowhere differentiable
    iii) The function is continuous and differentiable at a single point.

Unit 3:

E) Short type questions (Marks 5):

1. Define first and second kind of improper integrals.
2. Check for convergence of the following (each having 5 marks): i) $\int_1^\infty 1/x^p\,dx$, ii) $\int_0^\infty \exp(-ax)\ dx$, iii) $\int_0^1 1/x^p\ dx$, iv) $\int_0^\infty 1/(x^2+\sqrt{x})\ dx$, v) $\int_0^\infty 1/(e^x+1)\ dx$
3. State different properties of Gamma and Beta functions.

F) Broad type questions (Marks 10):

1. Study the convergence of Gamma function.
2. Study the convergence of Beta function.
3. Discuss different tests for convergence in connection with improper integral.

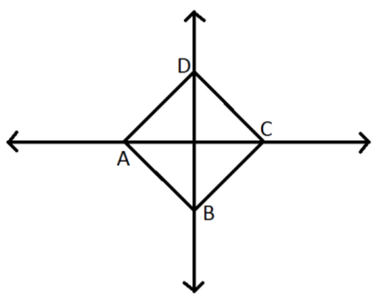
Unit 4:

G) Short type questions (Marks 5):

1. Discuss the theory of maximum-minimum of function with two variables.
2. Discuss the concept of polar transformation and find Jacobian of transformation for three variables.
3. What is saddle point of a function? Give an example to explain the concept.
4. Evaluate: $\int\int_{\mathcal{D}} \exp\left(-(x^2+y^2)\right) dxdy$, where $\mathcal{D}$ is the region between the two circles $x^2+y^2=1$ and $x^2+y^2=4$.

H) Broad question type (Marks 10):

1. Discuss the concept of Lagrange Multiplier and its use in finding the maximum-minimum of functions. Provide a suitable example.
2. i) Evaluate $\int\int_R \left(\frac{x-y}{x+y+2}\right)^2 dxdy$ over the region R pictured.
   Where $A\ (-1,0), B\ (0,-1), C\ (1,0)$ and $D\ (0,1)$.
   ii) Find the maximum value of the function $f(x,y)=x^2y$ when it is given that $x^2+y^2=3$.



CC-4 : Probability and Probability Distributions II

1. What is moment generating function (m.g.f.)? Derive the *m.g.f.* of the rectangular distribution with *p.d.f.* $f(x) = \frac{1}{(b-a)}; \quad a < x < b$.

2. Four coins, not all of which are unbiased, are thrown simultaneously and the numbers of heads are noted. If the probability of head for the four coins are 3/8, 5/8, 3/8 and 1/2 respectively, derive the probability generating function and hence find the probability of having two heads.

3. Let X be normal with mean zero and let Y be independent of X with $P(Y = 1) = P(Y = -1) = \frac{1}{2}$.
   (i) Find the distribution function of XY.
   (ii) Find the correlation coefficient between X and XY and comment on their dependence.

4. For the probability mass function
$$P[X = j] = \frac{a_j \theta^j}{f(\theta)}, \qquad j = 0, 1, 2, \dots\dots \; ; \; \theta > 0,$$
where $a_j \geq 0$ and $f(\theta) = \sum_{j=0}^{\infty} a_j \theta^j$, find the m.g.f. in terms of $f(\cdot)$.

5. The joint density of X and Y is given by $f(x, y) = \frac{\left(e^{-\frac{x}{y}} e^{-y}\right)}{y}; \; x > 0, y > 0$. Find $E(X|Y = y)$.

6. For $N(\mu, \sigma^2)$ distribution establish the following relation for the central moments
$$\mu_{2r+2} = \sigma^2 \mu_{2r} + \sigma^3 \frac{d\mu_{2r}}{d\sigma}.$$
Also, derive the *m.g.f.* of $N(\mu, \sigma^2)$ distribution. Use it to find the expected value of the geometric mean of two independent lognormal random variables.

7. When is a distribution said to be truncated? Let X follows the standard exponential distribution with mean θ and C be any positive constant. Find the mean and variance of X truncated for all values above C.

8. Define the negative binomial distribution. Find out its *m.g.f.* and hence find out its co-efficient of variation.

9. Let $X \sim N(\mu, \sigma^2)$. Find the density of X truncated on the left at a and on the right at b. For any positive c, if $a = \mu - c$ and $b = \mu + c$, then find the mean and variance of the truncated distribution.

10. Consider the random variable X with *p.d.f.* $f(x) = b\, exp[-b(x - a)], \; a < x < \infty$. Show that $E|X - \mu_e| = (log_e 2)/b$, where μ_e is the median of the distribution.

11. Show that the moment generating function of a Cauchy distribution does not exist.

12. Show that for a Gamma(k) distribution, Mode = Mean – $s.d./\sqrt{k}$. Also show that excess of kurtosis of the distribution is $6/k$.

13. Let X and Y be two independent random variables follow common geometric distribution. Find $Var(Y|X+Y=k)$.

14. Show that $\binom{k+x-1}{x}p^k q^x \rightarrow e^{-\mu}\frac{\mu^x}{x!}$ as $p \rightarrow 1$ and $k \rightarrow \infty$ in such a way that $kq = k(1-p)$ remains constant.

15. Let X be binomial random variable with parameters n and p. Obtain $E[(1+t)^X]$ and hence find $E(X)$. If Y be independently and identically distributed with X, show that the distribution of $X - Y$ is symmetric about zero.

16. If X is a Poisson random variable with parameter λ, then find the value of $E(1+X)^{-1}$.

17. Define probability generating function (*p.g.f.*) of a discrete random variable. Let $S_1 = X_1 + X_2$ and $S_2 = X_1 - X_2$ where X_i's are *i.i.d.* random variables each assuming the values $1, 2, \ldots, a$ with the same probability $1/a$. Find the probability generating function of S_1 and S_2. Hence determine the probability that X_1 exceeds X_2.

18. Show that in a series of $2s + 1$ trials with success probability 0.5, the most probable no. of success is s and the corresponding probability is

$$T_S = P(X = s) = \frac{1 \cdot 3 \cdot 5 \cdots (2s-1)}{2 \cdot 4 \cdot 6 \cdots 2s}.$$

Also show that $\frac{1}{2\sqrt{s}} < T_S < \frac{1}{\sqrt{2s+1}}$.

19. Show that the absolute mean deviation about mean of $Bin(n, p)$ distribution is

$$2m\binom{n}{m}p^m q^{n-m+1} = (2mpq/\pi)^{1/2} \quad \text{where } np < m \leq np + 1.$$

20. Let X be a Poisson variate with mean θ. Show that $E(X^2) = \theta E(X+1)$. If $\theta = 1$, show that $E(|X - \theta|) = \frac{2}{e}$.