DEPARTMENT OF MATHEMATICS
MAULANA AZAD COLLEGE

MathZiN

2024-2025

Volume--1

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Principal's Message

With great delight, I extend my warmest greetings to all members of our esteemed institution as we present the latest edition of our Mathematics Magazine. Witnessing the dedication and enthusiasm of our students and faculty in curating this publication fills me with immense pride.

Mathematics is more than just a subject; it is a universal language that reveals the mysteries of the universe and equips us with problem-solving skills that transcend academic boundaries. In this edition, you will discover a diverse range of articles, problems, and insights that highlight the multifaceted nature of mathematics and its applications across various fields.

I extend my heartfelt gratitude to the editorial team, comprised of both students and faculty members, for their tireless efforts in bringing this magazine to life. Their unwavering commitment to promoting mathematical knowledge and fostering a love for the subject within our community is truly commendable.

May this magazine serve as a source of inspiration, igniting curiosity and a passion for the endless possibilities within the realm of mathematics. I hope it not only deepens our understanding of the subject but also strengthens the bonds among us, fostering a sense of camaraderie.

Let us celebrate the achievements of our students and faculty, and continue to nurture a culture of learning and excellence in our institution. Thank you for your continued support, and I look forward to witnessing the ongoing growth and success of our mathematics community.

Principal
Maulana Azad College

Message from the Head of the Department Dr. Somnath Bandyopadhyay, Maulana Azad College, Kolkata

It is with immense pleasure that I extend a warm welcome to all of you to the vibrant realm of our College Mathematics Magazine. As the Head of the Mathematics Department, I am thrilled to witness the launch of this remarkable initiative that celebrates the beauty, diversity, and intellectual richness of the mathematical world.

Our College Mathematics Magazine is not just a publication; it is a testament to the passion, curiosity, and brilliance that define our mathematics community. Mathematics, as we know, is not merely a subject but a universal language that unveils the secrets of the universe. Through this magazine, we aim to cultivate a deeper appreciation for the elegance and transformative power that mathematics brings to our lives.

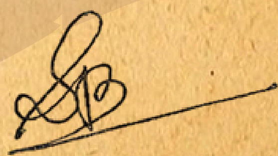
Within these pages, you will find a diverse collection of articles, features, and insights that highlight the extraordinary achievements of our students and faculty. This magazine serves as a platform to showcase the exceptional talent and creativity that thrive within our department.

I encourage each of you—whether a seasoned mathematician or someone just embarking on their mathematical journey—to actively engage with the content. This magazine is designed as a space for everyone to learn, explore, and be inspired by the boundless possibilities of mathematics.

I would like to extend my heartfelt gratitude to the dedicated team of students and faculty members whose hard work and enthusiasm have brought this publication to life. Your unwavering commitment to excellence is evident on every page, and I am confident that this magazine will be a source of pride and inspiration for our mathematics community.

As we embark on this exciting journey together, let us celebrate the beauty of mathematics and the intellectual curiosity that drives us to explore its infinite depths. May this magazine be a source of inspiration, motivation, and a shining example of the enduring spirit of the Mathematics Department at Maulana Azad College.

Wishing you all a fantastic experience as you delve into the pages of our inaugural Mathematics Magazine!



Head
Mathematics Department,
Maulana Azad College

TEACHER'S EDITORIAL

DR. NANDA DAS **DR.BABLI SAHA**

Dear Readers,

It is with great pleasure and enthusiasm that we welcome you to the latest issue of Mathematics Magazine. As we embark on this mathematical journey together, we are reminded of the profound beauty and significance that mathematics holds in our lives.

In this edition, we have curated a diverse collection of articles that traverse the expansive landscape of mathematics. From Algebra, Number Theory, and Real Analysis in pure mathematics to Mathematical Biology and Applied Physics, these contributions reflect the richness of the field. Our contributors—ranging from current and former students to esteemed members of our faculty—have crafted insightful pieces that we are confident will engage, inspire, and perhaps challenge your understanding of the mathematical universe.

Beyond the articles, this issue highlights various programs and extracurricular activities organized by the department. These initiatives showcase not only the depth of mathematical knowledge but also its profound impact on the world around us.

We extend our sincere gratitude to all contributors for sharing their expertise and passion for mathematics, and to our dedicated editorial team for their tireless efforts in bringing this issue to life.

We hope that Mathematics Magazine continues to serve as a source of intellectual stimulation and a catalyst for fostering a deeper appreciation of the mathematical sciences.

Happy reading!

Associate Professor,
Mathematics Department,
Maulana Azad College

Associate Professor,
Mathematics Department,
Maulana Azad College

STUDENT'S EDITORIAL

-A Journey Beyond Numbers

Dear Readers,

Welcome to this edition of our college Mathematics Magazine! As we dive into the vast ocean of mathematical wonders, we are reminded of the elegance, precision, and sheer beauty that mathematics offers. Often regarded as the most fundamental of all disciplines, mathematics serves as the backbone for subjects ranging from Physics and Chemistry to social sciences like Economics and modern fields such as Computing and Data Science.

In this issue, we embark on a journey beyond mere numbers, delving into the depths of mathematical concepts that not only shape our understanding of the world but also spark our imagination. This magazine is the result of the relentless effort and dedication of the students from the Mathematics Department, who have been skillfully guided and supported by our esteemed professors.

Our goal through this publication is to demonstrate that mathematics is far more than an abstract concept confined to textbooks and classrooms; it is a dynamic and powerful tool with countless real-world applications.

We, the student editors of MATHZIN, extend our heartfelt gratitude to all the students and professors of our department for their invaluable contributions. We hope you find this edition filled with fascinating insights and amazing facts. May the beauty of mathematics continue to inspire and captivate us all!

Best regards,

STUDENT'S CORNER

- The History And Development Of Zero

  --Masud Rana

- Sukuntala Devi & The Fascinating World Of Math Paradoxes

  --Ishika Roy

- The Future of Mathematics

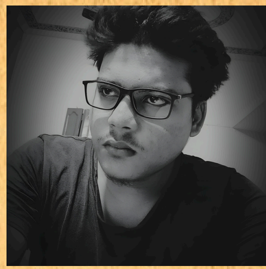
  --Dibyendu Dhali

- KNOT THEORY

  --Writam Bhattacharya

The History and Development of Zero

By Masud Rana



Zero's progress can be traced biologically across different cultures and epochs for a verylong time and in fact, it's one of the major pillars of modern mathematics. Uponestablishing where zero came from, one can effectively appreciate the manner in whichhumanity's understanding and embrace of mathematics changed ideologically.

Early Representations and Placeholders

- The history of zero and its early representation and the symbols associated with it areamong the key turning points in the history of mathematics and the numeral systems asthis shows the nature of the evolution of human understanding concerning numbersand the very important feature of some number systems, which is zero.

- **Babylonian Placeholder**

- About 300 B.C.E. a group of Babylonians existed that worked on a base 60 numbersystem and for that matter, during their time, they had problems differentiating "two"from "120" or "three thousand six hundred". Hence, they came up with the idea of adumb form marker which was an angled wedge. Even though this idea did help clarifynumerical representation, when it came to numbers, zero was never regarded as anumber. Instead, it was a role that served as an indication of position.

- **Mayan Civilization**
- Meanwhile, on the other side of the world, the Mayan civilization that resided in middle America independently constructed an advanced base number 20 system. In the fourth century A.D, they created a new symbol in shape of a shell that represented zero,primarily within their calendrical computations. This symbol enabled the Mayans toper form complex astronomical and chronological calculations, highlighting an advanced understanding of zero as a placeholder. However, similar to the Babylonian usage, it did not extend to a broader numerical or philosophical context.
- **Greek Hesitancy**
- Ancient Greek philosophers and mathematicians did not readily accept the use of the number zero. Their numerical system did not include a symbol for zero and the idea of nothing' was philosophically difficult. Such delays were due to metaphysical considerations regarding the issues of void, being and how these should be viewed which then made it difficult for Greeks to understand the concept of zero in mathematics.

The Indian Subcontinent: The Birthplace of Zero as a number

- The Indian sub-continent on the other hand came to view and use zero as a number. The advancement towards this shows the consistence and development in history of arithmetic and mathematics, which in turn helped modify the other civilizations that came to borrow from their numerals and scientific concepts.
- **Early Notation and Symbols**
- About 300 B.C.E. a group of Babylonians existed that worked on a base 60 number system and for that matter, during their time, they had problems differentiating "two"from "120" or "three thousand six hundred". Hence, they came up with the idea of adumb form marker which was an angled wedge. Even though this idea did help clarify numerical representation, when it came to numbers, zero was never regarded as anumber. Instead, it was a role that served as an indication of position.
- **Brahmagupta's Development**
- Important advancements in the mathematicians' works emerged in context of Brahmagupta's life when zero began to be referred to as a number by the mathematician. Brahmagupta's belief concerning the usage of zero can be found in hisbook Brahmasphuṭa siddhānta where he laid down the rules of arithmetic including zero, addition and subtraction.

- Epigraphic Evidence

- The Gwalior's Chaturbhuj Temple, India records one of the earliest known uses of the numeral of zero through an inscription that has been dated to be in the year 876 CE.This inscription bears witness to the use of zero in the built environment reinforcing its use in everyday life and administration.

Transmission to the Islamic World and Europe

- The very first transmission of mathematical knowledge in ties, especially the idea of 'zero' from the Islamic sphere to the European one was very essential in the evolution of Western mathematics and science.

- Transmission to Europe

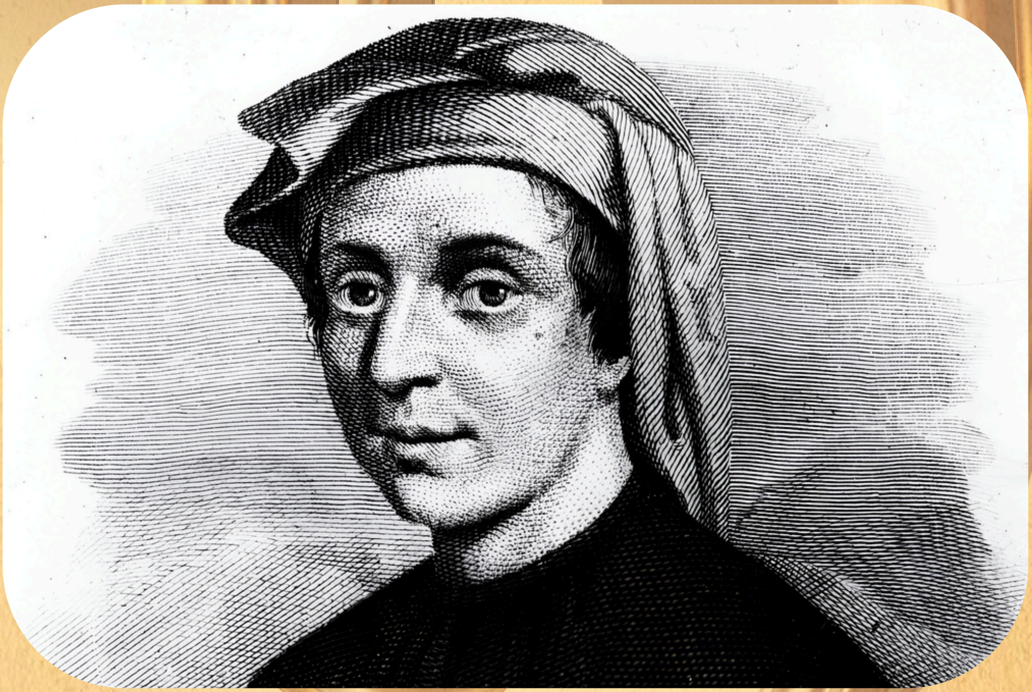
- The conduit through which this knowledge was transferred to the western portion of the world consisted mostly of many translations of Arabic mathematical works done in the medieval times. In particular, Europeans particularly in the twelfth century translated several significant Arabic works into their language. Al-Khwarizmi, Al-Khwarizmi Messier treatises were among these and his name even gave rise to the term"algorithm" pointing out his influence on mathematical processes.

Islamic Scholarship and the Adoption of Zero

- After its very basic invention in India, which dates back to around the 5th century BC, zero was further refined by Islamic scholars as late as the eighth and ninth centuries.The Persian mathematician Muhammad ibn Musa al-Khwarizmi, celebrated as the father of algebra, introduced the concept of zero to the Islamic world in his seminal work Kitabal-Jabr wa'l-Muqabala.

In the House of Wisdom in Baghdad, Al-Kwharizmi developed an Arabic numeric system with the number zero, called in Arabic 'sift'. Not only did this text help to build the foundations of Algebra, it contributed to the wider acceptance of zero too as one of the most important numbers in use today.

Fibonacci and the spread of zero worldwide



- Next on our historical journey is Fibonacci, also known as Leonardo of Pisa, who carried the torch of '0' and the Hindu-Arabic decimal system of Al-Kwarizmi, and brought it toEurope. Fibonacci learnt about '0' and decimal mathematics from Arab traders he met while accompanying his father on merchant tours in Tunisia. He immediately realised the superiority of the decimal system compared to previously used Roman numbers.This new type of mathematics spread to the rest of Europe through his book, Liber Abaci (Book of Calculation), published in 1202.

Relation of Aryabhata with Zero

- Though Aryabhata didn't denote zero by a symbol, the implication became clear in his work. His place-value system not only required a placeholder to denote an absent (zero)digit at a particular place but also indirectly explained how meaningless other digits would become without it. Zero was also introduced as a part of recipes necessary to solve quadratic and other indeterminate equations.However, formal definition and integration of zero in number systems can be credited to Indian mathematicians who lived after Aryabhata, particularly Brahmagupta (7thcentury), who explicitly denoted zero by a dot or small circle (another invention ofancient Indians) sometime around 650 A.D., as well as defined rules for arithmetic operations with zero. So while Aryabhata might've explained the philosophy behind use of null quantities, itwas the works of later Indian mathematicians which presented zero in its full glory and laid down the laws involving null quantities.

- ## Philosophical and Cultural Implications

- The invention of zero was as much a mathematical as it was a cultural and philosophical advancement. In many cultures, the concept of ‘nothingness’ held deep metaphysical import. Thus in ancient Greece, where debates raged about the physical reality of a void and the nature of existence, records indicate that the Greeks struggled with adopting zero as a number. In contrast, Indian philosophy with its elaborate concepts of ‘shunya’ or emptiness probably facilitated an easier acceptance of zero.

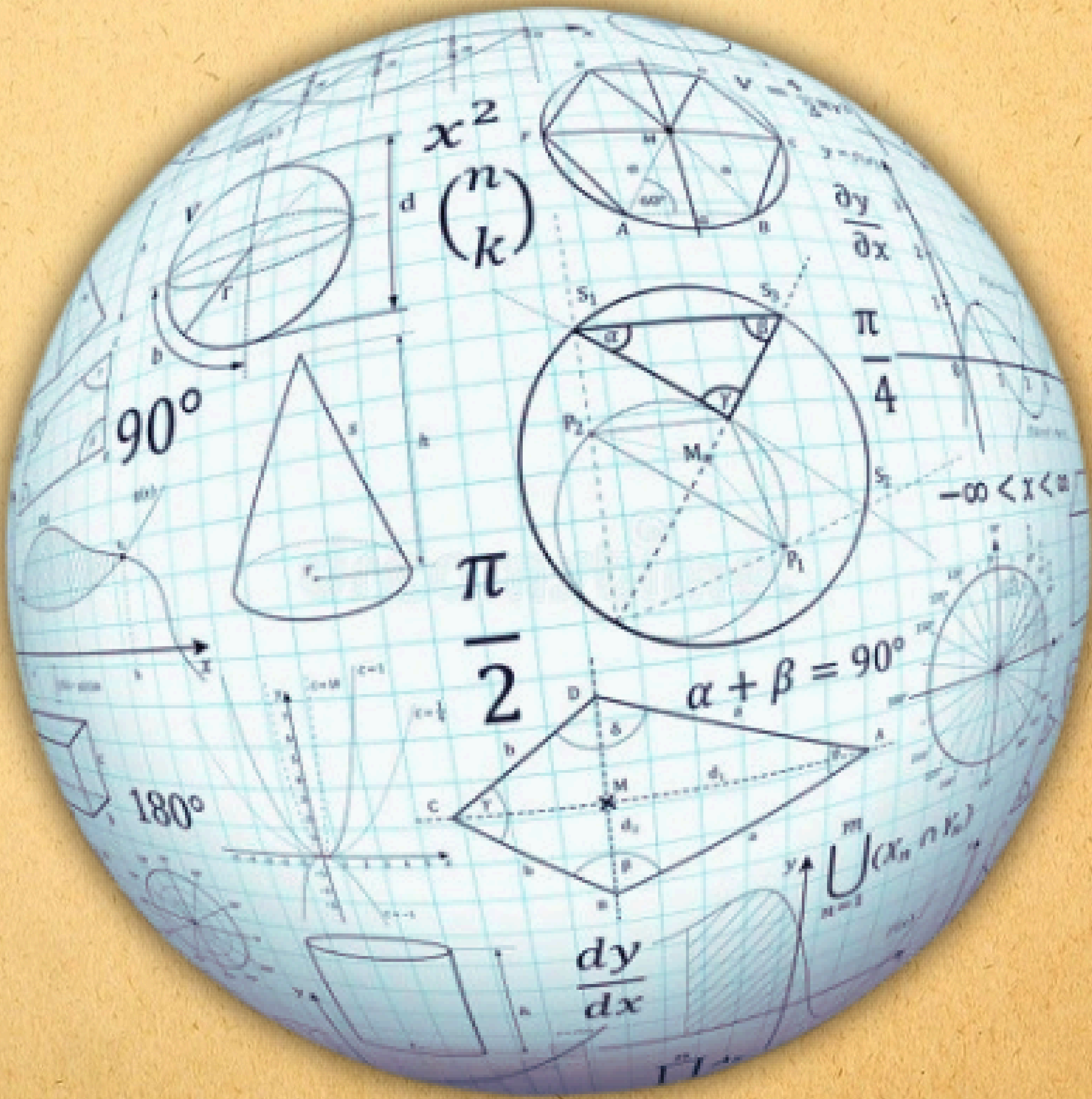
- ## Conclusion

- The development of zero was a transformative journey that reshaped mathematics and laid the ground work for modern science and technology. From its early use as a placeholder in ancient numeral systems to its recognition as an independent number in India, and its subsequent transmission to the rest of the world, zero's history underscores the dynamic interplay between mathematical innovation and cultural context.

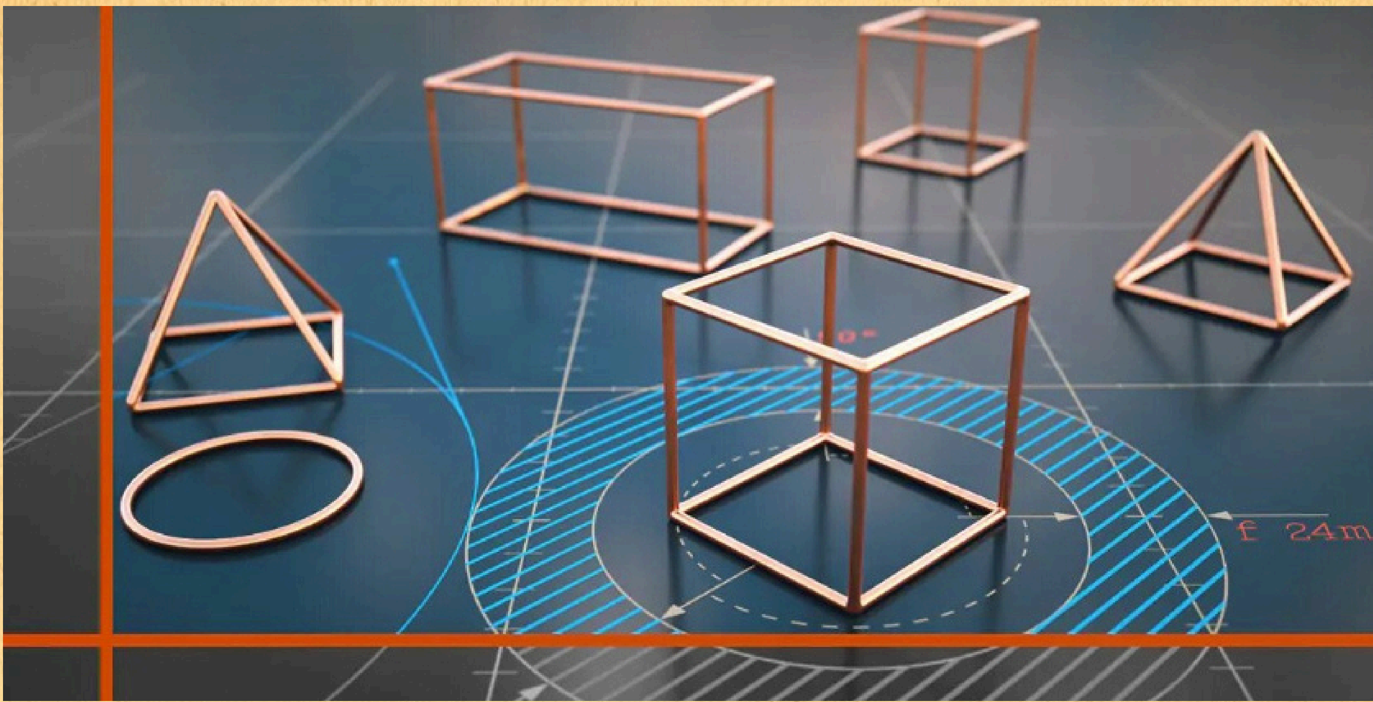
The Future of Mathematics

2024-2025

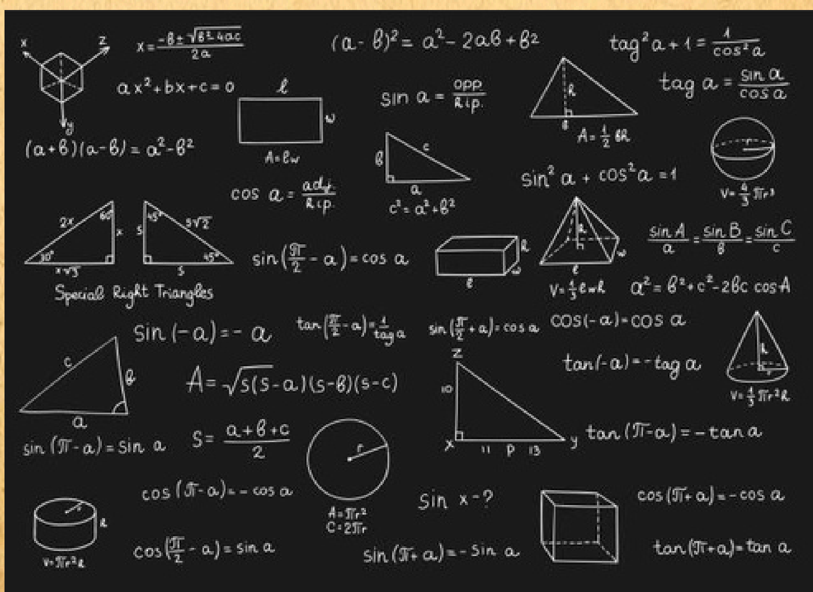
Futuristic Math



MATHEMATICAL MODELING FOR COMPLEX SYSTEM

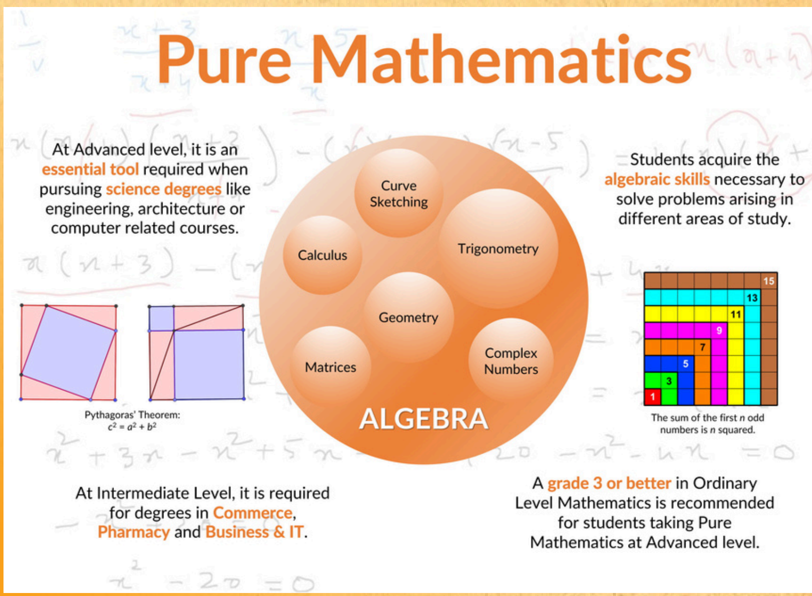


As computational power increases, the ability to model complex, real-world systems— from climate change to financial markets—will continue to expand. Mathematical models will be crucial in understanding and predicting phenomena such as pandemics, natural disasters, and economic crises. These models could become more sophisticated through the integration of artificial intelligence (AI) and machine learning.



Advances In Pure Mathematics

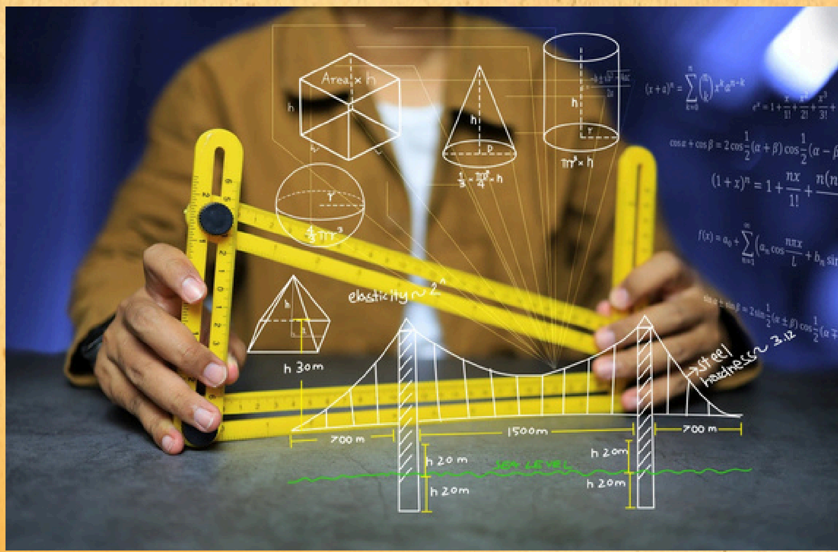
While applied mathematics may be more visible in the future, pure mathematics will continue to evolve as well. Fields like Number theory, algebra, topology, and geometry will explore deeper and more abstract concepts. The Future of pure mathematics will likely involves:



Abstract Mathematics: Despite its practical applications, pure mathematics —focused on understanding abstract concepts such as number theory, algebraic geometry, and differential topology —will continue to evolve. New branches of abstract, like homotopy type theory or noncommutative geometry, may lead to the discovery of mathematical structures with unforeseen applications.

Advances in Applied Mathematics

Applied Mathematics is the branch of Mathematics focused on using mathematical techniques and models to solve real-world problems across various fields, such as science, engineering, economics, and technology. Unlike pure mathematics, which deals with abstract concepts, applied mathematics aims to provide practical solutions to complex system and processes.



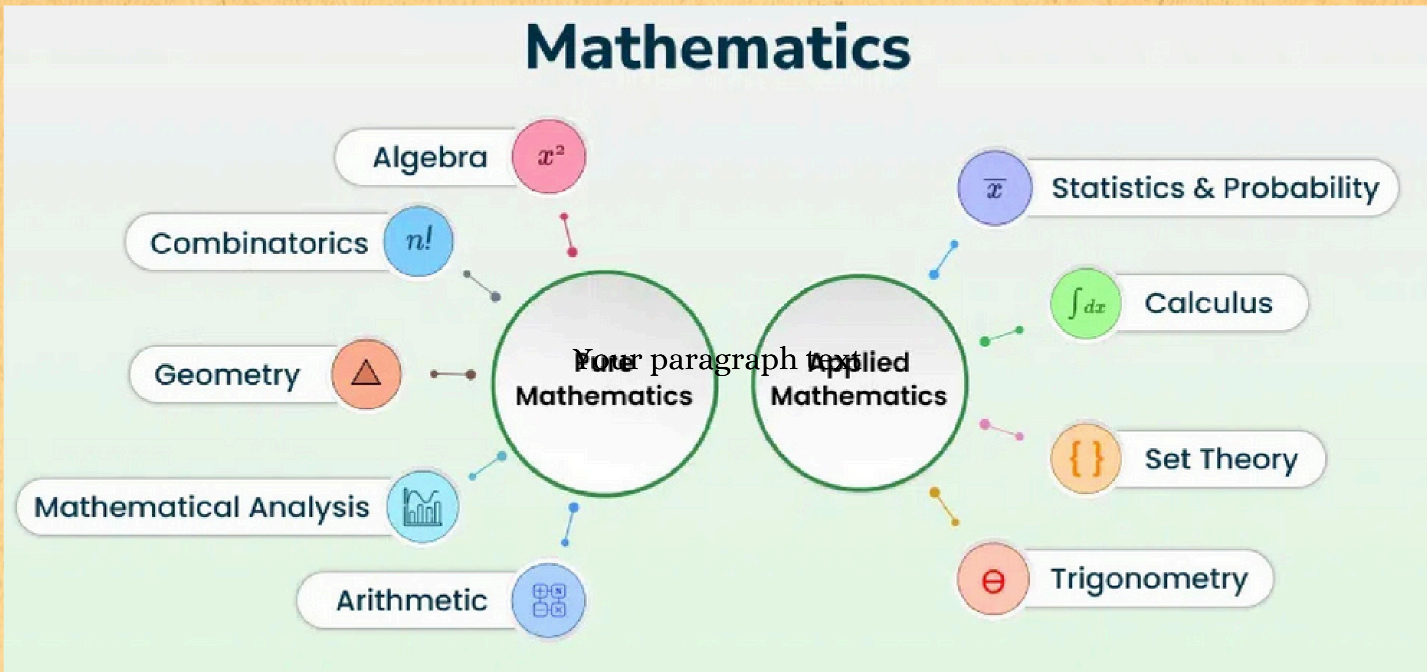
1.**Mathematical Modeling: Creating Mathematical representation of real-world phenomenon to predict and analyze their behavior.**

2. Numerical Analysis: Developing algorithms for solving complex mathematical problems through approximation methods.

3. Mathematical Finace: Using mathematics models to understand. and predict financial markets, optimize portfolios, and manage risk.

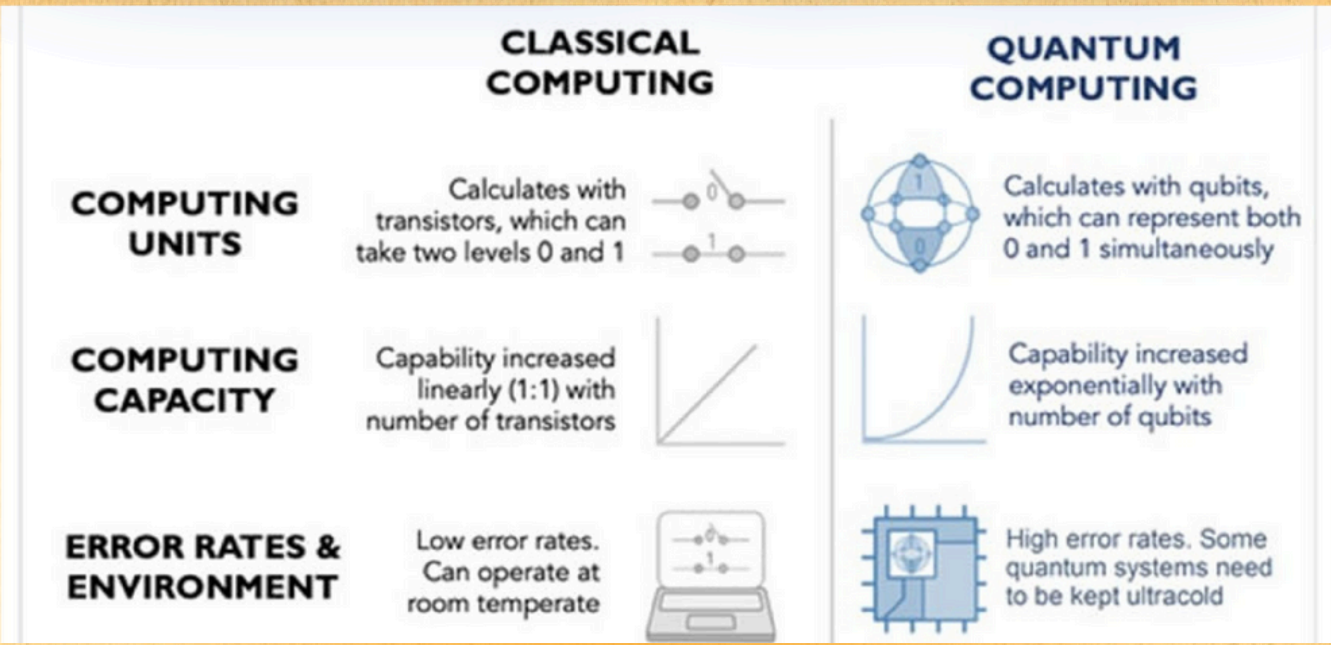
4. Differential Equation: Modeling dynamic system and understanding how quantities change over time or space, used in physics, biology, and engineering.

Comparison : Pure and Applied Mathematics



- **Applied Mathematics focuses on practical problem-solving and real-world applications. It uses mathematical techniques to address challenges in fields like engineering, physics, economics, and computer science. Its goal is often to create models, simulations, or algorithms that provide solutions to specific problems.**
- **Pure Mathematics is more abstract and theoretical. It emphasizes exploring mathematical concepts, structures, and theories for their intrinsic value, without immediate concern for practical applications. Topics include number theory, algebra, topology, and analysis.**

Quantam Computing.



- As computational tools continue to evolve, mathematics will increasingly focus on computational techniques and algorithm design,. The Future of Mathematics education and research will require students to master not only abstract theories but also computational methods that can handle real world data. This includes:

i) Numerical Methods
ii) simulation and modeling techniques
iii) Optimization algorithm
iv) Machine Learning
v)Software tools like Matlab, python and Mathematica

In essence, the computational aspect of Mathematics will become as central as theoretical work, merging the two point a more practical, applied discipline.

Future Career Possibilities in Mathematics

Mathematics offers a vast range of career opportunities due to its critical role in problem-solving, analysis, and logical reasoning. Some key possibilities include:

- **1. Academia and Research: Teaching at schools, colleges, or universities and conducting research in pure or applied mathematics.**
- **Data Science and Analytics: Analyzing large datasets to inform decisions in business, healthcare, and technology.**
- **Actuarial Science: Assessing risk for insurance companies, banks, or investment firms.**
- **Engineering and Technology: Working on innovations in fields like robotics, AI, and software development.**
- **Finance and Economics: Quantitative analysis in banking, investment, and stock market predictions.**
- **Cryptography: Developing secure communication systems for cybersecurity.**
- **Biostatistics: Applying mathematical models to study biological or medical data.**

WHERE CAN MATHS TAKE YOU?

ENGINEERING

42% of the engineering workforce in the UK is over the age of 45. This means there will be a huge demand for young engineers in the decades to come!

CAREER PATHS

- Chemical Engineer
- Civil Engineer
- Mechanical Engineer

IT & THE INTERNET

People with qualifications in Information Technology have one of the highest rates of employment in the UK.

CAREER PATHS

- Games Developer
- Software Programmer
- Network Engineer
- Web Designer

ACCOUNTANCY

The number of accountancy associations in the UK has grown by 3.7% since 2006. Student numbers have been growing even more quickly!

CAREER PATHS

- Tax Accountant
- Auditor
- Forensic Accountant

SCIENCE & RESEARCH

It's predicted that in the next few years, 1 in 4 jobs will have been created by science and research - leading to 140,000 new science jobs by the end of 2018.

CAREER PATHS

- Research Scientist
- Mathematician
- Statistician

BANKING & FINANCE

51% of employers in the banking and finance industry believe there is a skills shortage amongst their employees.

CAREER PATHS

- Retail Banker
- Financial Advisor
- Fund Manager
- Stockbroker

CONSULTANCY

In the future, consultants with skills and knowledge in digital technology, financial services, retail and infrastructure will be in high demand.

CAREER PATHS

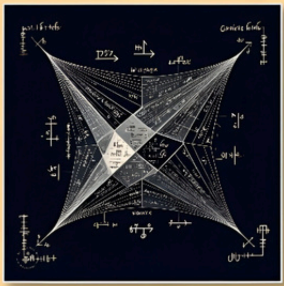
- Management Consultant
- Data Analyst
- IT Consultant

Sukuntala Devi
& The Fascinating World of Math Paradoxes

Ishika Roy
SEM-1, 2024-28

Sukuntala Devi, known as the "Human Computer", astonished the world with her ability to solve massive mathematical problems in seconds. Her incredible skills spark curiosity about the mysteries of math, leading us to explore a fascinating topic: paradoxes—statements that seem impossible but are true. Just like Sukuntala Devi's extraordinary abilities, paradoxes make us question what we think we know.

What Are Paradoxes?



A paradox in math is like a puzzle that doesn't make sense at first glance but holds a deeper truth. It challenges our usual way of thinking and makes us realize that logic can sometimes feel like magic. Sukuntala Devi's unique way of solving problems—bypassing traditional methods—reminds us of how paradoxes break our expectations.

Historical Origins of Paradoxes

Paradoxes have been around for a long time, starting with ancient Greek thinkers like Zeno, who used paradoxes to question how we understand things like motion and time. Later, philosophers like Socrates used paradoxes to challenge people's thinking on various topics. In modern times, paradoxes helped shape the development of logic and mathematics, with famous examples like Russell's Paradox and Gödel's Incompleteness Theorems showing that some things can't be explained within existing systems, pushing us to think deeper about logic and truth.

The Philosophy Behind Paradoxes

The philosophy behind paradoxes is rooted in challenging our understanding of logic, reasoning, and reality. Paradoxes often expose contradictions or unexpected truths that force us to rethink fundamental concepts. Philosophers use paradoxes to explore the limits of knowledge and the complexities of concepts like time, infinity, truth, and existence.

For example, Zeno's paradoxes challenge the idea of motion and change by suggesting that an infinite number of steps are needed to reach a destination, yet we clearly experience movement. Similarly, paradoxes like the liar paradox ("This statement is false") show how language and truth can be self-contradictory. Through paradoxes, philosophy questions whether our assumptions about the world are as solid as we think, opening up deeper discussions about the nature of reality, perception, and logic.

Famous Math Paradoxes That Feel Like Magic

1. The Infinite Ball Puzzle (*Banach-Tarski Paradox*)**:**

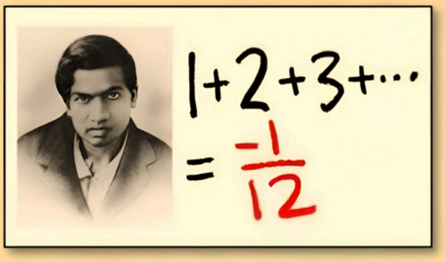
Imagine taking a ball, cutting it into a few pieces, and then rearranging them to create two balls of the same size. This is mathematically possible! It's as mind-boggling as Sukuntala Devi's ability to solve gigantic problems in seconds. Here's how it works: this is a result of the Banach-Tarski paradox, a fascinating concept in set theory and geometry. It's a reminder of how counterintuitive infinite mathematics can be! This paradox shows the sheer depth of mathematical abstraction, where logic allows us to explore realms beyond physical reality.

2. The Never-Ending Race (*Zeno's Paradox*)**:**

Picture a race where a tortoise gets a head start, and a faster runner tries to catch up. The runner keeps covering smaller and smaller distances but never quite reaches the tortoise. It's a paradox that feels as impossible as Sukuntala Devi's instant calculations. This is known as Zeno's paradox, highlighting the perplexities of infinite division in motion. Mathematics shows us that logic can sometimes defy our everyday intuition!

3. The Crazy Infinite Sum (*Ramanujan's Paradox*)**:**

Mathematician Srinivasa Ramanujan found that the sum of all natural numbers is −1/12-\frac{1}{12}−121—a result that sounds absurd but is true in advanced physics. Similarly, Sukuntala Devi's results could seem unbelievable but were always accurate. This counterintuitive result appears in string theory and quantum physics, showing the deep connections between mathematics and the universe. Sometimes, the most surprising answers lead to the most profound discoveries!



4. The Missing Dollar Paradox:

Three people pay $30 for a hotel room, splitting it equally. Later, the manager realizes the room costs $25 and gives $5 back. The bellboy, confused, keeps $2 and gives $1 to each person. Each paid $9, totaling $27, plus the $2 with the bellboy—where's the missing dollar? This playful paradox highlights how our logic can be tricked. The trick lies in how the situation is framed—the $27 already includes the $25 room cost and the $2 with the bellboy.

Does a Paradox Really Exist?

Do paradoxes truly exist, or are they just a result of our limited understanding? A paradox challenges what seems impossible, revealing deeper layers of reality we haven't fully grasped yet. While they may not exist in a traditional sense, they push us to rethink our understanding of the world.

Why Paradoxes Matter

Paradoxes teach us to think beyond the obvious. They remind us that math is not just a subject—it's a way of exploring the mysteries of the universe. Sukuntala Devi's incredible mind showed us that math can be playful, magical, and full of surprises, just like paradoxes.

They challenge us to question assumptions and embrace curiosity. In doing so, they reveal the beauty and depth of logical thinking!

Role of Paradoxes in Advancing Mathematics

Paradoxes have played an important role in moving mathematics forward by pointing out problems and limitations in current theories. For example, Russell's Paradox showed that there were flaws in the way sets (groups of things) were being understood, which led to the creation of better and more precise ways to deal with sets, like Zermelo-Fraenkel set theory. Another example is Cantor's work on infinite sets, which helped mathematicians understand that not all infinities are the same size. By challenging the way things were thought about, paradoxes pushed mathematicians to improve their definitions, sharpen their logic, and rethink basic ideas, which in turn advanced the field of mathematics.

Connections Between Paradoxes and Infinity

Paradoxes and infinity go hand in hand because infinity often leads to some pretty mind-bending ideas.

For example, **Cantor's Paradox** shows that not all infinities are equal—some are bigger than others, like the difference between counting whole numbers and real numbers. Then there's the **Banach-Tarski Paradox**, which says you could take a sphere, cut it into pieces, and somehow end up with two identical spheres. It sounds impossible, but it works in math **Zeno's Paradoxes** mess with our understanding of motion by breaking it into infinite parts, making us question how infinity really works. These paradoxes reveal just how strange & puzzling infinity can .

Sukuntala Devi's Connection to Paradoxes

While Sukuntala Devi didn't directly work on paradoxes, her life was full of them.

She solved problems faster than computers, challenged the limits of the human brain, and made the impossible seem possible. For example, her ability to calculate the 23rd root of a 201-digit number faster than a computer feels as paradoxical as the Banach-Tarski puzzle. She also showed us that math isn't just about formulas—it's about imagination and creativity. Like paradoxes, her methods were surprising, intuitive, and groundbreaking.

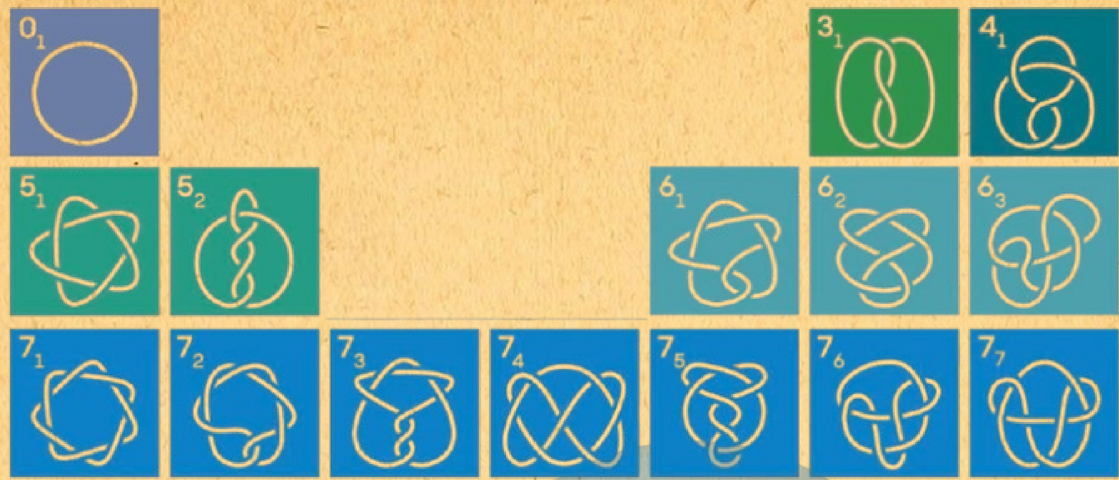
Inspiration for the Future

Sukuntala Devi's extraordinary story and the captivating world of paradoxes teach us to see math as more than formulas and equations—it's a journey of exploration and wonder. They remind us that the limits we perceive are often just illusions, waiting to be overcome with curiosity and creativity.

Paradoxes challenge us to think deeply, question assumptions, and embrace the unknown. Similarly, Sukuntala Devi showed us that with determination and passion, the impossible becomes achievable. Whether you're unraveling the mysteries of a paradox or pursuing your biggest dreams, there's always more to learn, discover, and achieve. The journey never ends—it only grows richer with every step forward.

KNOT THEORY

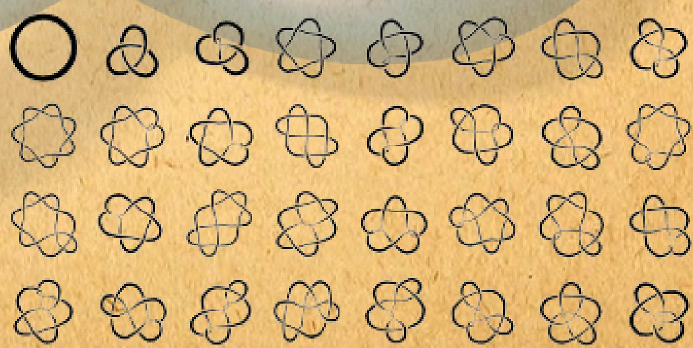
--by Writam Bhattacharya



- In the late 19th and early 20th centuries,mathematicians such as William Thomson (LordKelvin) and James Clerk Maxwell began studying knots from a mathematical perspective.Knot theory is a captivating branch of mathematics that delves into the intricate properties of knots and links in three-dimensional space. By examining the topological characteristics of these knots, mathematicians can uncover profound insights into the fundamental nature of space and matter. The study of knots has far-reaching implications,influencing fields such as physics, biology, computerscience, and materials science. From understanding the behavior of subatomic particles to analyzing the structure of DNA, knot theory provides a unique lens through which to explore the complexities of our universe.

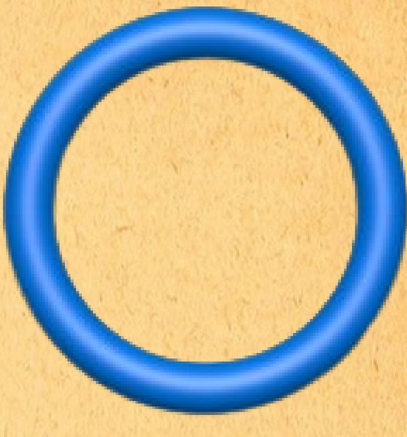
AN UNKNOT

The Prime Knots



BASIC KNOTS

- 1. Trivial Knot (Unknot): A knot with no crossings.
- 2. Trefoil Knot: A knot with three crossings, and the simplest non-trivial knot.
- 3. Figure-Eight Knot: A knot with four crossings, and the second-simplest non-trivial knot.

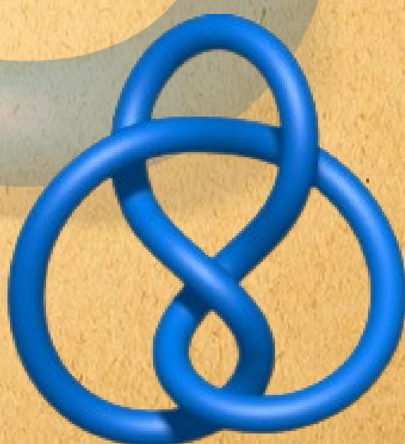


TORUS KNOTS

- 1. Torus Knot: A knot that can beembedded on the surface of a torus (doughnut-shaped surface).
- 2. (2,3)-Torus Knot: A torus knot with two twists and three turns.

SATELLITE KNOTS

- 1. Satellite Knot: A knot that contains another knot as a connected sum.
- 2. Connect Sum: A knot formed by joining two knots together.



HYPERBOLIC KNOTS

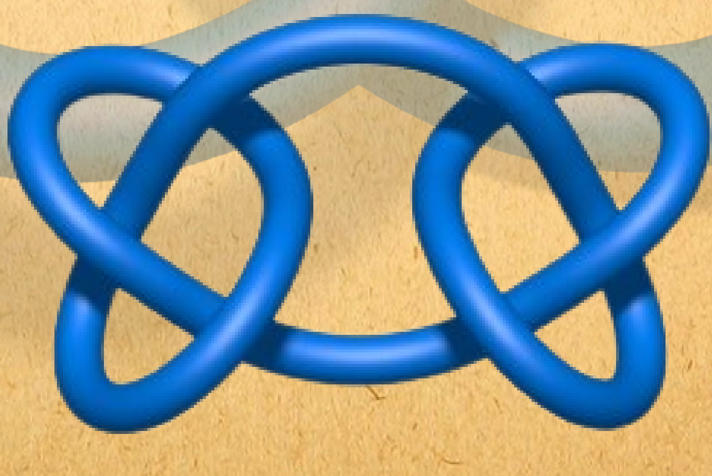
1. Hyperbolic Knot: A knot that has a hyperbolic structure.
2. Figure-Eight Knot (Hyperbolic): The figure-eight knot is a hyperbolic knot.

- A knot is called prime if it can not be represented as a connected sum of two knots such that both of these are knotted. Any knot which is not prime is called composite.

INVARIANTS

In knot theory, an invariant is a property or quantity that remains unchanged underdifferent representations or transformations of a knot. In other words, an invariant is a characteristic of a knot that does not depend on how the knot is drawn or manipulated.

COMPOSITE KNOT



CONTRIBUTIONS OF MATHEMATICIANS IN KNOT THEORY

1. Carl Friedrich Gauss (1777-1855): Gauss is considered one of the founders of knot theory. He studied the properties of knots and introduced the concept of the "linking number".
2. William Thomson (Lord Kelvin) (1824-1907)proposed the "Vortex Theory" of atoms, which led to the study of knots and links in physics.
3. James Clerk Maxwell (1831-1879) developed the concept of "topological invariants" to study the properties of knots and links.
4. Henri Poincaré (1854-1912) introduced the concept of "homotopy" to study the properties of knots and links.
5. Emmy Noether (1882-1935) developed the "Noether's Theorem" which has implications for knot theory and its connections to physics.
6. James Waddell Alexander (1888-1971): Alexander introduced the Alexander polynomial, a fundamental invariant in knot theory.
7. Emmy Noether (1882-1935): Noether's work on abstract algebra laid the foundation for the study of knot groups and other algebraic invariants.
8. Kurt Reidemeister (1893-1971): Reidemeister introduced the Reidemeister moves, which are used to classify knots and links.

Applications of Knot Theory

1. Physics: Knot theory has applications in quantum field theory, string theory, and condensed matter physics.
2. Biology: Knot theory has applications in the study of DNA topology and protein structure.
3. Computer Science: Knot theory has applications in computer graphics, robotics, and network topology.

CONCLUSION

Knot theory is a rich and fascinating field that has evolved significantly over the centuries. From its humble beginnings in topology and geometry to its modern applications in physics,biology, and computer science, knot theory has proven to be a versatile and powerful tool forunder standing complex systems.

Alumni's Corner

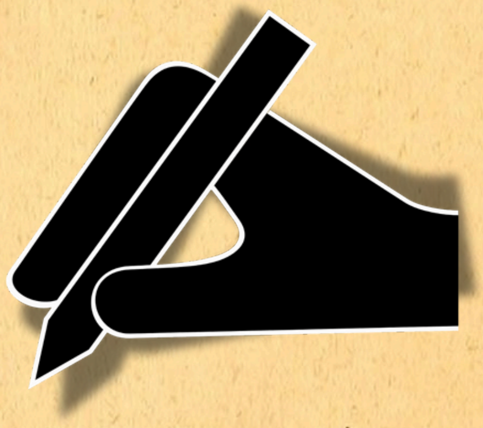


IMAGE RECONSTRUCTION USING NONSMOOTH CONVEX OPTIMIZATION

Arghya Sinha

1. INTRODUCTION

Image reconstruction is the process of creating or improving an image when the original data is incomplete, blurry, or damaged. This is applicable in various fields, such as:

- **Medical imaging**: Creating clear 3D images of the human body, such as CT or MRI scans, from raw measurement data.
- **Astronomy**: Enhancing images of stars and planets captured through telescopes by reducing noise and distortions.
- **Everyday applications**: Fixing blurry photos or sharpening low-resolution images.

The goal of image reconstruction is to take messy or incomplete data and turn it into a clear, accurate image. This process often involves two key steps:

1. **Matching the data**: Ensuring the reconstructed image aligns with the measured or observed data.
2. **Using prior knowledge**: Applying assumptions about the image, such as smoothness, sharp edges, or patterns common to similar images.

As a first step, we consider the following model where we want to recover an unknown image $\boldsymbol{\xi} \in \mathbb{R}^n$ from linear measurements

$$\boldsymbol{b} = \mathbf{A}\boldsymbol{\xi} + \boldsymbol{\eta}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the forward operator, $\boldsymbol{b} \in \mathbb{R}^m$ is the observed image, and $\boldsymbol{\eta} \in \mathbb{R}^m$ is white Gaussian noise. The forward operator varies depending on the application and is typically assumed to be known in advance. For instance, when attempting to deblur an image blurred due to poor focus, a Gaussian blur linear operator can be used as the forward operator. The question is: how can we recover $\boldsymbol{\xi}$ if $\mathbf{A}$ and $\mathbf{b}$ are already known? The answer is that, in most cases, exact recovery of $\boldsymbol{\xi}$ is not possible. Instead, we aim to find an approximate reconstruction, whose quality depends on the method used.

(a) *Forward Operator ($\mathbf{A}$)* **(b)** *Clear Image ($\boldsymbol{\xi}$)* **(c)** *Observed ($\boldsymbol{b}$)*

Fig. 1: *An example of a forward operator and the observed image. The first figure on the left shows a blur operator applied to a single white pixel. The image on the right is the observed blurry image, which is the result of applying the forward operator to the original clear image.*

2. SMOOTH OPTIMIZATION

The process of image reconstruction can be approached by solving an optimization problem where a reconstruction $\boldsymbol{x}$ is obtained by minimizing a well-defined loss function. A common choice for such a loss function is $f(\boldsymbol{x}) = \frac{1}{2}\|\mathbf{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2$, which measures the discrepancy between the observed data $\boldsymbol{b}$ and the reconstructed image $\mathbf{A}\boldsymbol{x}$. Formally, this leads to the following optimization problem:

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}). \tag{2}$$

This particular formulation is advantageous due to the smooth and convex nature of the objective function $f(\boldsymbol{x})$. These mathematical properties enable the use of

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efficient optimization techniques, such as the gradient descent method [1], to solve (2). Gradient descent is an iterative algorithm that updates the current estimate $\boldsymbol{x}_k$ using the gradient of f. Specifically, starting from an initial estimate $\boldsymbol{x}_0$, the iterations are given by:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \gamma \nabla f(\boldsymbol{x}_k), \tag{3}$$

where $\gamma > 0$ is the step size (learning rate), and $\nabla f(\boldsymbol{x})$ denotes the gradient of the loss function.

The convexity of f guarantees convergence of the sequence $\{\boldsymbol{x}_k\}$ to the global minimum, provided the step size γ is chosen appropriately. This makes gradient descent a reliable and widely used method in optimization problems of this kind.

However, in practical scenarios, the problem in (2) is often ill-conditioned, which can lead to numerical instabilities or solutions that are not meaningful in the context of image reconstruction. To solve such issues, we apply a regularizer term with the loss function. Incorporating a regularizer transforms the problem into a more robust formulation, allowing the optimization process to better handle challenges associated with noise, incomplete data, or poorly conditioned systems. Various regularization techniques, such as Tikhonov regularization or sparsity-promoting norms, have proven effective in improving reconstruction outcomes and are widely explored in image processing and optimization.

3. REGULARIZERS FOR IMAGE RECONSTRUCTION

We now discuss three commonly used regularizers: Tikhonov regularization, Lasso, and Total Variation (TV).

3.1. Tikhonov Regularization

Tikhonov regularization, also known as ridge regularization, penalizes the ℓ_2-norm of the solution to enforce smoothness and reduce sensitivity to noise. The regularized optimization problem is formulated as:

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \left\{ f(\boldsymbol{x}) + \lambda \|\boldsymbol{x}\|_2^2 \right\}, \tag{4}$$

where $f(\boldsymbol{x})$ is the loss function, $\lambda > 0$ is the regularization parameter, and $\|\boldsymbol{x}\|_2^2 = \sum_{i=1}^n x_i^2$ is the squared ℓ_2-norm. The parameter λ controls the trade-off between fidelity to the data and smoothness of the solution. Tikhonov regularization is particularly effective when the desired solution is expected to be smooth or when dealing with noisy measurements.

3.2. Lasso Regularization

Lasso regularization promotes sparsity in the solution by penalizing the ℓ_1-norm. The corresponding optimization problem is:

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \left\{ f(\boldsymbol{x}) + \lambda \|\boldsymbol{x}\|_1 \right\}, \tag{5}$$

where $\|\boldsymbol{x}\|_1 = \sum_{i=1}^n |x_i|$ is the ℓ_1-norm. Lasso is particularly useful in applications where the solution is expected to be sparse, meaning that many elements of $\boldsymbol{x}$ are zero or near-zero. This makes Lasso widely used in compressed sensing and feature selection tasks.

3.3. Total Variation (TV) Regularization

Total Variation (TV) regularization is designed to preserve edges in the reconstructed image by penalizing the total variation of the solution. The TV norm is defined as:

$$\|\boldsymbol{x}\|_{\text{TV}} = \sum_{i=1}^{n-1} |x_i - x_{i+1}|, \tag{6}$$

The regularized optimization problem is:

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \left\{ f(\boldsymbol{x}) + \lambda \|\boldsymbol{x}\|_{\text{TV}} \right\}. \tag{7}$$

TV regularization is particularly effective in preserving sharp edges and piecewise-smooth structures, making it a popular choice in image denoising and deblurring tasks.

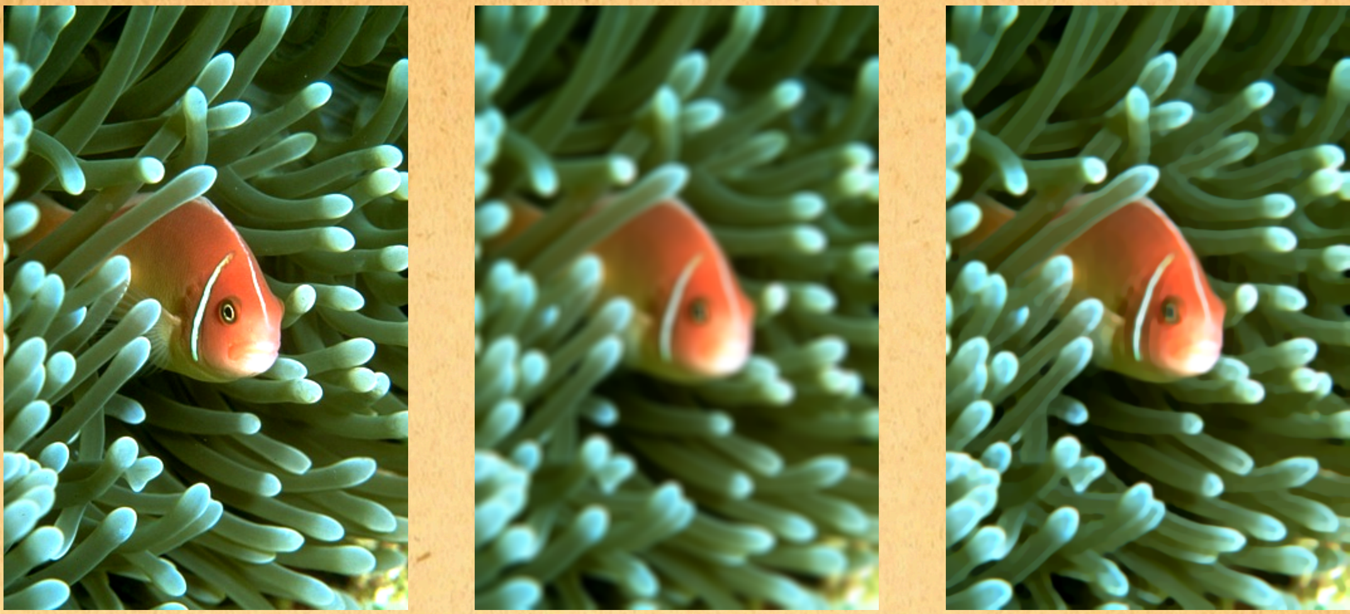
3.4. Choosing the Regularizer

The choice of regularizer depends on the characteristics of the problem and the prior information available about the solution. For example, Tikhonov regularization is suitable for smooth solutions, Lasso is ideal for sparse solutions, and TV is preferred when preserving edges is critical. The regularization parameter λ must also be carefully tuned to achieve a balance between data fidelity and the imposed regularization constraint. Other than these classical options, there are other modern denoiser based regularizations that work well in practice [2] by incorporating prior knowledge into the reconstruction.

4. PROXIMAL GRADIENT METHOD FOR REGULARIZED OPTIMIZATION

The optimization problems involving regularization can be expressed as:

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \left\{ f(\boldsymbol{x}) + g(\boldsymbol{x}) \right\}, \tag{8}$$



(a) *Clear Image* **(b)** *Blurry Image* **(c)** *Reconstruction*

Fig. 2: *Reconstruction of a blurry image using TV regularizer in* (7)

where $f(\boldsymbol{x})$ is a smooth, differentiable function with a Lipschitz-continuous gradient, and $g(\boldsymbol{x})$ is a convex function that acts as the regularizer, which may be non-smooth. As highlighted in the previous section, while the Tikhonov regularizer is smooth, other regularizers such as Lasso or Total Variation (TV) are inherently non-smooth. This non-smoothness prevents the direct application of gradient descent to solve (8). Instead, a more versatile approach known as the Proximal Gradient Method is employed. This method combines gradient descent for the smooth term $f(\boldsymbol{x})$ with a proximal operator for the non-smooth term $g(\boldsymbol{x})$.

4.1. Proximal Operator

The proximal operator of a convex function $g(\boldsymbol{x})$ is defined as:

$$\text{prox}_{\lambda g}(\boldsymbol{z}) = \arg\min_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ \frac{1}{2}\|\boldsymbol{x}-\boldsymbol{z}\|_2^2 + \lambda g(\boldsymbol{x}) \right\}, \quad (9)$$

where $\lambda > 0$ is a step size or regularization parameter. Intuitively, the proximal operator computes a point $\boldsymbol{x}$ that balances proximity to $\boldsymbol{z}$ and the regularization imposed by $g(\boldsymbol{x})$.

4.2. Proximal Gradient Iterations

The proximal gradient method performs the following iterative updates:

$$\boldsymbol{x}_{k+1} = \text{prox}_{\gamma g}\left(\boldsymbol{x}_k - \gamma\nabla f(\boldsymbol{x}_k)\right), \quad (10)$$

where $\gamma > 0$ is the step size. Here, the gradient descent step $\boldsymbol{x}_k - \gamma\nabla f(\boldsymbol{x}_k)$ minimizes the smooth term $f(\boldsymbol{x})$, while the proximal operator $\text{prox}_{\gamma g}$ enforces the regularization defined by $g(\boldsymbol{x})$.

4.3. Applications to Regularizers

The regularizers often has a closed form solution for the proximal operator. This makes Proximal-Gradient method very effecient.

- **Tikhonov Regularization**: For $g(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}\|_2^2$, the proximal operator simplifies to a scaled identity mapping:

$$\text{prox}_{\lambda g}(\boldsymbol{z}) = \frac{\boldsymbol{z}}{1+\lambda}. \quad (11)$$

- **Lasso Regularization**: For $g(\boldsymbol{x}) = \|\boldsymbol{x}\|_1$, the proximal operator corresponds to the soft-thresholding operator:

$$\text{prox}_{\lambda g}(\boldsymbol{z})_i = \text{sign}(z_i)\max\{|z_i| - \lambda, 0\}, \quad \forall i. \quad (12)$$

- **Total Variation (TV) Regularization**: For $g(\boldsymbol{x}) = \|\boldsymbol{x}\|_{\mathrm{TV}}$, the proximal operator requires solving a more complex subproblem. Efficient algorithms, such as those proposed in [3], are often employed for this purpose.

4.4. Convergence and Benefits

The proximal gradient method is guaranteed to converge to a global minimum under mild conditions, such as convexity of $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ and appropriate choice of the step size γ. This method is particularly attractive for large-scale problems because it exploits the structure of $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$, allowing for efficient computations. Additionally, the proximal gradient method can handle a wide variety of regularizers, making it a versatile tool for solving regularized optimization problems.

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Subgroups of $(\mathbb{R}, +)$

-By Bishyay Majumdar

UG: MAULANA AZAD COLLEGE
YEAR OF ADMISSION: 2019
YEAR OF COMPLETION THE UG COURSE: 2022

We know that, the real line $\mathbb{R}$ with the usual addition forms a commutative group. The aim of this short write up is to categorize any subgroup of the group $(\mathbb{R}, +)$ with the help of the usual topological structure of $(\mathbb{R}, +)$.

Let us first state some simple definitions and facts:

Topological Group: A group $(G,\cdot)$ with a topology τ on G is called topological group if the two group operations $M: G \times G \to G$ defined by $M(x,y) = x \cdot y$ and $i: G \to G$ defined by $i(x) = x^{-1}$ are continuous or equivalently the map $(x,y) \to x \cdot y^{-1}$ is continuous.

Fact 1: For any topological group $(G,\cdot,\tau)$ (or simply G for the sake of simplicity) and $g \in G$, left translation $L_g: G \to G$, defined by $L_g(x) = g \cdot x$ is a homeomorphism. Right translation $R_g: G \to G$ defined by, $R_g(x) = x \cdot g$ is also a homeomorphism of G.

Fact 2: The inversion $i: G \to G$ defined by $i(x) = x^{-1}$ is a homeomorphism of G.

Examples: Clearly $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}\backslash\{0\},\cdot)$, $(\mathbb{R}^n, +)$ are all topological groups. Any group equipped with the discrete topology is a topological group.

(N.B.: A reader, who does not accustomed with the notion of arbitrary topological space so far, can simply think of the point set topology of the real line and group operation as the usual addition, homeomorphism as continuous bijection with continuous inverse (or simply a map that preserves the topological properties) to understand the topic and can proceed to the next theorem.)

Theorem: Any subgroup of $(\mathbb{R}, +)$ with usual topology is either dense in $\mathbb{R}$ or cyclic.

Proof: Let H be a subgroup of the additive group $\mathbb{R}$. If, H is cyclic, then there is nothing to prove.
Let, H is non cyclic subgroup of $\mathbb{R}$.
Let, if possible, H be not dense in $\mathbb{R}$.
Then, there exists an interval $(a,b) \subset \mathbb{R}$ such that $\mathbb{R} \cap (a,b) = \emptyset$. Let $\epsilon = \frac{b-a}{2}$. Then clearly, $0 \in (-\epsilon,\epsilon) \cap H$. Now, if, $t \in (-\epsilon,\epsilon) \cap H\backslash\{0\}$, then $|t| \in H$ (since, $t > 0 \Rightarrow |t| = t \in H$ and $t < 0 \Rightarrow |t| = -t \in H$, H being a subgroup) and by Archimedean property of real line if n is the largest integer for which $n|t| \leq a$, then $a < (n+1)|t| < a + \frac{b-a}{2} < b$, a contradiction to the fact that $H \cap (a,b) = \emptyset$, since H being a subgroup, contains $(n+1)|t|$.
Now, H being non trivial (since it is non cyclic), let $x(> \epsilon) \in H$. Now, let $\mathcal{A} \coloneqq \{(i\epsilon, (i+2)\epsilon) : i \in \mathbb{R}^+)\}$. Then $\mathcal{A}$ is an open cover of the set $[\epsilon, x]$, and $[\epsilon, x]$ being compact, $\mathcal{A}$ has a finite sub

cover. Also, each element of $\mathcal{A}$ can contain at most one element of H [since $(-\epsilon, \epsilon) \cap H = \{0\}$ and each interval in $\mathcal{A}$ is nothing but translation of $(-\epsilon, \epsilon)$ by $(i+1)\epsilon$]. Thus, we can say there exists a least positive element in H say k. Now, for any $h \in H$, there exist an integer m (by Archimedean property of real numbers) such that, $mk \leq h < (m+1)k \Rightarrow 0 \leq h - mk < k$. Since, $h, k \in H$, so, $h - mk \in H$, which implies that $h - mk = 0$ (since, k is the smallest positive integer in H). Hence, $h = mk$, for some $m \in \mathbb{Z}$.
Thus, we have that $H =< k >$ i.e; H is a cyclic group, which contradicts our hypothesis. Thus, H must be dense in $\mathbb{R}$. (Proved).

By the last theory, let us now make some observations below.

Observation 1: $(\mathbb{R}, +)$ can not contain any non-trivial subgroup of finite order.

Proof: In fact, by the above theorem, if H be finite subgroup in $\mathbb{R}$, then it is not dense and hence is cyclic. Let, $H =< p >$. Then, there is $n \in \mathbb{Z}^+$, such that $np = 0$, which implies $p = 0$, i.e. $H = \{0\}$, that is, H is trivial.

Observation 2: Every proper subgroup of $(\mathbb{R}, +)$ is either dense or closed in $\mathbb{R}$.

Proof: Let, H be a non dense subgroup of $\mathbb{R}$. Then, H is cyclic infinite group and hence isomorphic to $\mathbb{Z}$. Thus, $H = x\mathbb{Z}$, for some $x \in \mathbb{R}\backslash\{0\}$. Since, $\phi: \mathbb{R} \to x\mathbb{R}$, defined by $\phi(a) = a \cdot x$, is homeomorphism (since, $x \neq 0$), so, $x\mathbb{Z} = \phi(\mathbb{Z})$ is closed in $\mathbb{R}$, since so is $\mathbb{Z}$ in $\mathbb{R}$.

Observation 3: Any closed proper subgroup of $\mathbb{R}$ is endowed with discrete topology.

Proof: Clearly, the proof follows from the above observation.

Observation 4: For the additive group $(\mathbb{C}, +)$ or$(\mathbb{R}^n, +)(n \geq 2)$, the theorem does not hold.

Proof: In fact, taking $(\mathbb{R} \times \{0\}, +)$, or $(\mathbb{R}^{n-1} \times \{0\}, +)$ serves our purpose.

Before making any further observations let us prove a result:

Lemma: If A is compact and B is closed subset in $\mathbb{R}$, then $A + B \coloneqq \{a + b: a \in A, b \in B\}$ is closed subset of $\mathbb{R}$.

Proof: Let (x_n) be a sequence in $A + B$ converging to x in $\mathbb{R}$. We now show that, $x \in A + B$. Now, $x_n = a_n + b_n$ (say), where $a_n \in A$, and $b_n \in B$ for all $n \in \mathbb{N}$. Now, $x_n - b_n = a_n$ is a sequence in the compact metric space A and thus have a convergent subsequence, say, a_{r_n}converging to a in A. Then, $b_{r_n} = x_{r_n} - a_{r_n}$ is a convergent sequence converging to b (say). Now, B being closed, $b \in B$. Hence, $x = a + b \in A + B$, i.e. $A + B$ is closed in $\mathbb{R}$.

The above lemma is also true for more general set up, i.e. for any topological group and can be proved very similarly using the concept of 'Net', in place of sequence.

Observation 5: The above Lemma doesn't hold if both A and B are closed.

Proof: In fact, taking $\mathbb{Z}$ and $\alpha\mathbb{Z}$, where α is any irrational number, we get that, $\mathbb{Z} + \alpha\mathbb{Z}$ is a non cyclic subgroup of $\mathbb{R}$. Since, $\mathbb{Z} + \alpha\mathbb{Z} = \beta\mathbb{Z} \Rightarrow 1 \in \beta\mathbb{Z}$ and $\alpha \in \beta\mathbb{Z}$. So, $1 = \beta m$ and $\alpha = \beta n$, for some $m, n \in \mathbb{Z}$ which implies $\alpha = \frac{n}{m}$, which contradicts that α is irrational. Thus, $\mathbb{Z} + \alpha\mathbb{Z}$ being non cyclic subgroup of $\mathbb{R}$, is dense in $\mathbb{R}$. Also, clearly $\mathbb{Z} + \alpha\mathbb{Z}$ is properly contained in $\mathbb{R}$ and hence is not closed.

N.B: By the above observation we can conclude for any irrational α the subgroup $\mathbb{Z} + \alpha\mathbb{Z}$ of $\mathbb{R}$ is dense in $\mathbb{R}$.

Uncountable Proper Dense Subgroups of $(\mathbb{R}, +)$:

Let us conclude this small write up on the additive real group by giving some examples of uncountable dense proper subgroup of $\mathbb{R}$.

Example 1: We know that the dimension of the vector space $\mathbb{R}$ over $\mathbb{Q}$ is uncountable. So taking any uncountable subset of the basis and the if L be the vector subspace generated by that basis, then $(L, +)$ is uncountable (and hence non cyclic i.e. dense) subgroup of $(\mathbb{R}, +)$.

Example 2: Let, $H = \{x : \lim\limits_{n\to\infty} \sin(n!\,\pi x) = 0\}$. Then, clearly H is a subgroup of $\mathbb{R}$ under addition. Now, let $S = \{\sum_{i=0}^{\infty} \frac{a_i}{i!} : a_i \in \{0,1\}\}$. Clearly, then S is uncountable. Now, for any $x \in S$, we can say $n!\,x = \left(n!\,a_0 + \frac{n!}{1!}a_1 + \cdots + a_n\right) + \sum_{i=(n+1)}^{\infty} \frac{a_i}{i(i-1)\ldots(n+1)} = A_n + B_n$, where A_n is an integer. Since, $|B_n| \le \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots = \frac{1}{n}$, so, we have that, $\lim\limits_{n\to\infty} B_n = 0$, since, so is $\lim\limits_{n\to\infty} \frac{1}{n}$. $Now,\ \sin(A_n\pi + B_n\pi) = \sin(A_n\pi)\cos(B_n\pi) + \sin(B_n\pi) cos(A_n\pi)$

$$= 0 + \sin(B_n\pi)\cos(A_n\pi)$$

Since, $\cos(A_n\pi)$ is bounded and sin is continuous, so we have, $\lim\limits_{n\to\infty} \sin(A_n\pi + B_n\pi) = 0$ i.e. $\lim\limits_{n\to\infty} \sin(n!\,\pi x) = 0$, for all $x \in S$. Hence, we have H is an uncountable subgroup of $(\mathbb{R}, +)$.

A Brief Introduction to Central Simple Algebras

Sayan Pal

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In this article we will discuss some basic ideas about central simple algebras over a field k. It is known that the set of all square matrices $M_n(k)$ of order n forms an algebra with respect to the matrix multiplication with identity as I_n. This algebra has a center isomorphic to the field $k(\simeq k.I_n)$ and doesn't contain any two-sided ideal except $\{0\}$ and itself. We will see these algebras are the basic examples of central simple algebras. We will assume all the algebras here are finite dimensional and contains 1. We refer to ([**AM**]) for some basics on tensor products, ideals, rings and modules.

1 Introduction

Let k be any field and by an extension K/k we will mean a field containing k as a subfield. If A is an algebra defined over k, we write A_K for the K-algebra obtained by extending the scalars to K, i.e, $A_K = A \otimes K$. We also define the *opposite algebra* A^{op} by $A^{op} = \{a^{op} : a \in A\}$ with the operations defined as follows:

$$a^{op} + b^{op} = (a+b)^{op},\ a^{op}b^{op} = (ba)^{op},\ \alpha.a^{op} = (\alpha.a)^{op}$$

for $a, b \in A$ and $\alpha \in k$.

Definition 1.1. A *central simple algebra*(CSA) over a field k is a finite dimensional algebra $A \neq \{0\}$ with center $k(= k.1)$ which has no proper two-sided ideal except $\{0\}$. A central simple algebra A is called *division algebra* if every non-zero element in A is invertible.

As we can see the above definition has been motivated by the algebras $M_n(k)$ which we mentioned earlier. But these are not division algebras as we know all linear transformations defined on a vector space of dimension n are not invertible. Now we describe some properties of these algebras which help us to check whether an algebra is CSA or not.

2 Properties of CSA

The structures of CSA's are determined by the following well-known result due to Wedderburn:

Theorem 2.1. For an algebra A over a field k the following conditions are equivalent:

1. A is CSA.
2. There is a field K containing k such that $A_K \simeq M_n(K)$, for some n.
3. If K is an algebraically closed field containing k, $A_K \simeq M_n(K)$ for some n.
4. There is a finite dimensional central division algebra D over k and an integer r such that $A \simeq M_r(D)$.
5. The canonical map $A \otimes_k A^{op} \to End_k(A)$ which associate to $a \otimes b^{op}$ the linear map $x \mapsto axb$ is an isomorphism.

The fields for which condition (2) holds are called *splitting fields* of the algebra A. A central simple algebra A is called *split* if it is isomorphic to a matrix algebra $M_n(k)$. For example, every CSA over the field of complex numbers splits. One can check easily that the dimensions of the central simple algebras are always of the form n^2 for some n. $dim_k(A) = n^2$ if $A_K \simeq M_n(K)$ for some extension K/k. The integer n is called the *degree* of the algebra A and denoted by $deg(A)$. The degree of the division algebra D in condition (4) is called the *index* of A.

Definition 2.1. Let A and B be two finite dimensional CSA over k. They are called *Brauer-equivalent* if $M_l(A) \simeq M_m(B)$ for some integers l and m.

We can see that every CSA is Brauer-equivalent to one and only one division algebra. If we denote by $Br(k)$ the set of all Brauer equivalence classes of CSA's over k then tensor product gives us a group structure on $Br(k)$, called the *Brauer group* of k. The unit element in this group is the class of k which represents the class of matrix algebras over k. The inverse of the class of a CSA, A is the class of A^{op}, which follows from condition (5) of theorem 2.1. An interesting fact about these algebras is their automorphisms which we will describe next. The following result is due to Skolem and Noether:

Theorem 2.2. Let A be a CSA over k and $\rho : A \to A$ be an automorphism. Then $\exists\ a \in A$ which is invertible and $\rho(b) = aba^{-1}$, for all $b \in A$.

The automorphisms of the above type are called *inner-automorphisms*. So, every automorphism of a CSA is inner and from this we can identify all automorphisms of the matrix algebras $M_n(k)$ for any n.

3 Over Rings

Let R be a commutative ring with identity 1. Let A be an R-algebra which is a faithfully projective module as well (see [**AM**]). Then we define A to be an *Azumaya Algebra* if the canonical map $A \otimes_R A^{op} \to End_R(A)$ is an isomorphism of R-algebras. As we can see this is a generalization of the concept of CSA over a commutative ring R with identity. There are several similarities between the properties of CSA over fields and Azumaya algebras over rings. There are many structural differences as well between these two types of algebras. For a comparative study of these two algebras we refer the reader to ([**MJ**], [**PG**]).

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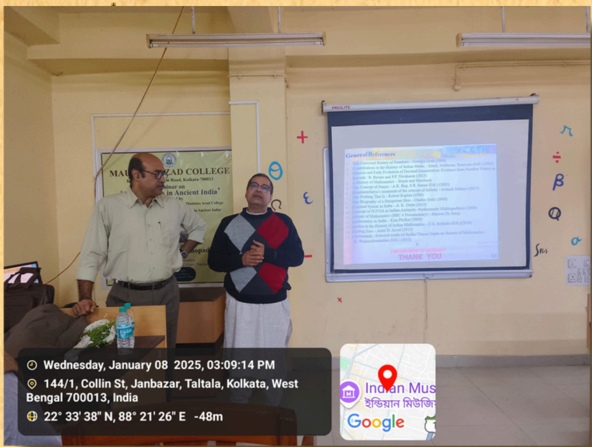
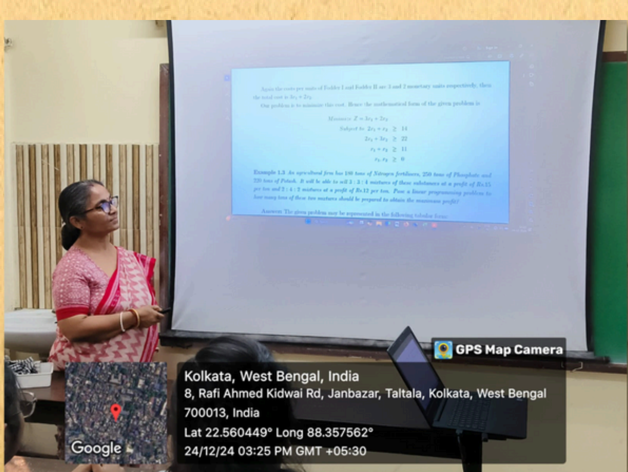
[PG] Central Simple Algebras and Galois Cohomology: Philippe Gille.

Picture Gallery



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